

Section 7.8 Improper Integrals.

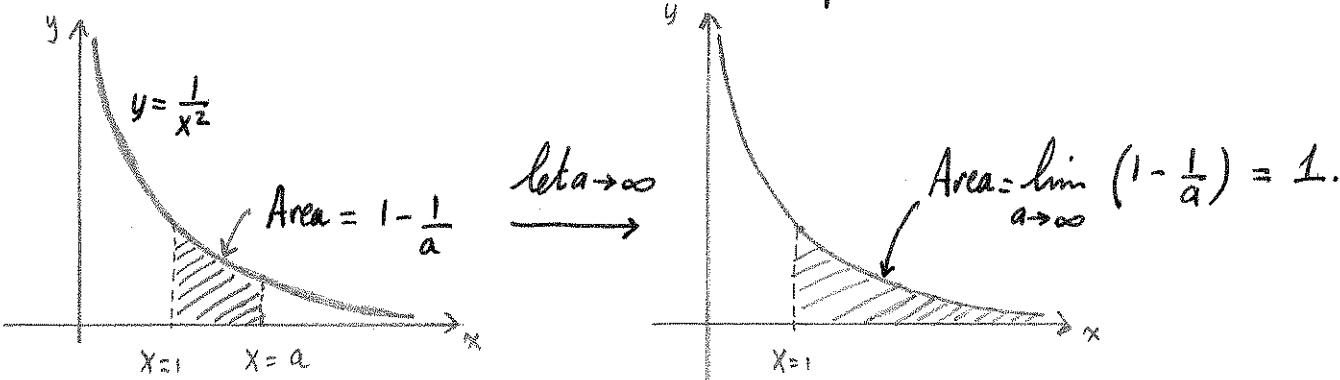
Goal: evaluate $\int_a^b f(x) dx$ where f may have a discontinuity on $[a,b]$, or where the endpoints a and/or b are infinite.

Type 1: Infinite intervals.

Suppose we're to find the area of the region between the graph of $y = \frac{1}{x^2}$ and the x -axis, to the right of $x=1$; that is, we would like to evaluate the definite integral $\int_1^\infty \frac{1}{x^2} dx$. The interval $[1, \infty)$ is infinite; we instead consider the finite interval $[1, a]$, where $a > 1$, then evaluate $\int_1^a \frac{1}{x^2} dx$. Finally, we think about what happens to this definite integral as a approaches infinity. We have

$$\int_1^a \frac{1}{x^2} dx = -\frac{1}{x} \Big|_1^a = 1 - \frac{1}{a}; \text{ this is the area between } x=1 \text{ and } x=a.$$

$$\text{Now let } a \rightarrow \infty; \lim_{a \rightarrow \infty} \int_1^a \frac{1}{x^2} dx = \int_1^\infty \frac{1}{x^2} dx = \lim_{a \rightarrow \infty} \left(1 - \frac{1}{a}\right) = 1.$$



In general,

- (a) if $\int_a^t f(x) dx$ exists for all $t \geq a$, then $\int_a^\infty f(x) dx = \lim_{t \rightarrow \infty} \int_a^t f(x) dx$, provided this limit exists (as a finite number).
- (b) if $\int_t^b f(x) dx$ exists for all $t \leq b$, then $\int_{-\infty}^b f(x) dx = \lim_{t \rightarrow -\infty} \int_t^b f(x) dx$, provided this limit is finite.

(C) $\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^a f(x) dx + \int_a^{\infty} f(x) dx$ provided the integrals on the right are finite. Here, any real number a can be used.

• Integrals of the form $\int_{-\infty}^b f(x) dx$ and $\int_a^{\infty} f(x) dx$ are called convergent if the limits exist, and divergent otherwise.

Example ① is $\int_1^{\infty} \frac{1}{x} dx$ convergent or divergent?

$$\int_1^{\infty} \frac{1}{x} dx = \lim_{t \rightarrow \infty} \int_1^t \frac{1}{x} dx = \lim_{t \rightarrow \infty} (\ln|x| \Big|_1^t) = \lim_{t \rightarrow \infty} (\ln(t) - \ln(1)) = \infty.$$

Therefore the integral in question is divergent. Note that this means that the area enclosed between $y = \frac{1}{x}$ and the x -axis, to the right of $x=1$, is infinite!

② Evaluate $\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx$. We split this integral at $x=0$; That is,

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = \int_{-\infty}^0 \frac{1}{1+x^2} dx + \int_0^{\infty} \frac{1}{1+x^2} dx. \text{ We evaluate each integral:}$$

$$\begin{aligned} \cdot \int_{-\infty}^0 \frac{1}{1+x^2} dx &= \lim_{t \rightarrow -\infty} \int_t^0 \frac{1}{1+x^2} dx = \lim_{t \rightarrow -\infty} \tan^{-1}(x) \Big|_t^0 = \lim_{t \rightarrow -\infty} (\tan^{-1}(0) - \tan^{-1}(t)) \\ &= 0 - (-\frac{\pi}{2}) = \frac{\pi}{2}. \end{aligned}$$

$$\cdot \int_0^{\infty} \frac{1}{1+x^2} dx = \lim_{t \rightarrow \infty} \int_0^t \frac{1}{1+x^2} dx = \lim_{t \rightarrow \infty} (\tan^{-1}(t) - \tan^{-1}(0)) = \frac{\pi}{2} - 0 = \frac{\pi}{2}. \text{ Therefore}$$

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

Theorem: $\int_1^{\infty} \frac{1}{x^p} dx$ is convergent if $p > 1$, and divergent if $p \leq 1$.

Type 2: discontinuous integrands. What if $f(x)$ has a discontinuity at the endpoints, or at a point c in between?

(a) if f is continuous on $[a, b]$, discontinuous at b , then approach b from the left: $\int_a^b f(x) dx = \lim_{t \rightarrow b^-} \int_a^t f(x) dx$.

(b) if f is continuous on $(a, b]$, discontinuous at a , then approach a from the right: $\int_a^b f(x) dx = \lim_{t \rightarrow a^+} \int_t^b f(x) dx$.

(c) if f is discontinuous at some c , where $a < c < b$, then we can write $\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$, assuming the integrals on the right are convergent.

Example ③ Evaluate $\int_0^1 \frac{1}{2\sqrt{x}} dx$. Here, $f(x) = \frac{1}{2\sqrt{x}}$ is discontinuous at $x=0$.

Therefore, $\int_0^1 \frac{1}{2\sqrt{x}} dx = \lim_{t \rightarrow 0^+} \int_t^1 \frac{1}{2\sqrt{x}} dx = \lim_{t \rightarrow 0^+} (\sqrt{x} \Big|_t^1) = \lim_{t \rightarrow 0^+} (\sqrt{1} - \sqrt{t}) = 1$.

④ Evaluate $\int_0^3 \frac{1}{x-2} dx$; Here, $f(x) = \frac{1}{x-2}$ has a discontinuity at $x=2$.

We split the interval $[0, 3]$ at $x=2$, and compute $\int_0^2 \frac{dx}{x-2}$ and $\int_2^3 \frac{dx}{x-2}$.

$$\cdot \int_0^2 \frac{1}{x-2} dx = \lim_{t \rightarrow 2^-} \int_0^t \frac{1}{x-2} dx = \lim_{t \rightarrow 2^-} (\ln|x-2| \Big|_0^t) = \lim_{t \rightarrow 2^-} (\ln|t-2| - \ln|-2|) = -\infty$$

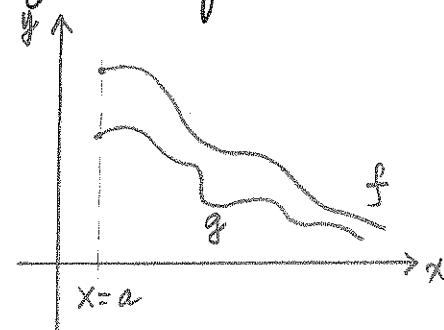
Since this integral is divergent, the original integral is also divergent, and we don't need to evaluate $\int_2^3 \frac{1}{x-2} dx$.

A comparison test for improper integrals: some improper integrals are difficult to compute; so we may instead compare them to easier improper integrals.

Theorem: suppose f and g are continuous fns, with $f(x) \geq g(x) \geq 0$ for all $x \geq a$.

then (a) if $\int_a^\infty g(x) dx$ diverges, so does $\int_a^\infty f(x) dx$;

(b) if $\int_a^\infty f(x) dx$ converges, so does $\int_a^\infty g(x) dx$.



Example ⑤ Show that $\int_1^\infty e^{-x^2} dx$ is convergent. Notice that, for $x \geq 1$, $-x^2 \leq -x$, and so $e^{-x^2} \leq e^{-x}$. We know $\int_1^\infty e^{-x} dx = \lim_{t \rightarrow \infty} \int_1^t e^{-x} dx = \lim_{t \rightarrow \infty} (-e^{-t} - (-e^{-1})) = \frac{1}{e}$, which is finite. Since $\int_1^\infty e^{-x} dx$ converges, and $e^{-x^2} \leq e^{-x}$ for all $x \geq 1$, and e^{-x^2}, e^{-x} are positive for all $x \geq 1$, then $\int_1^\infty e^{-x^2} dx$ is also convergent, by the comparison theorem.

⑥ Is $\int_1^\infty \frac{7}{2x^{1/2} + e^{5x}} dx$ convergent or divergent? For all $x \geq 1$, $\frac{7}{2x^{1/2} + e^{5x}} \leq \frac{7}{e^{5x}}$, and $\int_1^\infty \frac{7}{e^{5x}} dx = \lim_{t \rightarrow \infty} \left(-\frac{7}{5} e^{-5x} \right) \Big|_1^t = \frac{7}{5}$ (finite). By the comparison test, the original integral is convergent.

The limit comparison test: suppose f and g are positive, continuous functions on $[a, \infty)$, and $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = L$, where L is finite, and nonzero ($0 < L < \infty$). Then $\int_a^\infty f(x) dx$ and $\int_a^\infty g(x) dx$ both converge or both diverge.

Example ⑦ Show $\int_5^\infty \frac{6}{\sqrt{x-4}} dx$ diverges; we choose $g(x) = \frac{6}{\sqrt{x}}$. Then $\frac{6}{\sqrt{x-5}}, \frac{6}{\sqrt{x}} > 0$ for all $x \geq 5$, and $\lim_{x \rightarrow \infty} \frac{\frac{6}{\sqrt{x-4}}}{\frac{6}{\sqrt{x}}} = \lim_{x \rightarrow \infty} \sqrt{\frac{x}{x-4}} = 1$, and $0 < 1 < \infty$. Also $\int_5^\infty \frac{6}{\sqrt{x}} dx = \int_5^\infty 6 \cdot x^{-1/2} dx$ diverges ($p = \frac{1}{2} < 1$). By the limit comparison test, $\int_5^\infty \frac{6}{\sqrt{x-4}} dx$ also diverges.